PARTIAL DIFFERENTIAL EQUATIONS

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5. Nonlinear parabolic PDE and the Navier-Stokes equations

(1) Let $u(x,t)$ be the solution of the heat equation in $\mathbb{R}^n$

$$\begin{cases} \partial_t u - \Delta u &= 0 \qquad \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= u_\circ(x) \qquad \text{for} \quad t=0. \end{cases}$$

given by

$$u(x,t) = 1/(4\pi t)^{n/2} \int_{\mathbb{R}^n} u_\circ(y) e^{-\frac{|x-y|^2}{4t}} dy$$

Prove that, for any $k \in \mathbb{N}$, if $u_\circ \in C^k(\mathbb{R}^n)$ and all its derivatives (up to order k) are bounded then

$$\|u(t)\|_{C^{k+1}(\mathbb{R}^n)} \le \frac{C}{t^{1/2}} \|u_\circ\|_{C^k(\mathbb{R}^n)}$$

for some constant C depending only on n and k.

(3 points)

(2) Let $u, v \in C^\infty(\overline{\Omega} \times [0,T])$ be two solutions of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u &= f(u) \qquad \text{in} \quad \Omega \times (0,T] \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times [0,T] \end{cases}$$

with $f \in C^\infty$. Prove that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \le e^{Ct} \|u(0) - v(0)\|_{L^2(\Omega)}$$

for some constant C.

(3 points)

(3) (i) Let $u \in C^\infty(\overline{\Omega} \times (0,T))$ be a solution of

$$\begin{cases} \partial_t u - \Delta u &= f(u) \qquad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T) \end{cases}$$

Prove that

$$\frac{d}{dt} \int_\Omega \left(\tfrac{1}{2} |\nabla u|^2 - F(u) \right) = - \int_{\mathbb{R}^n} |\partial_t u|^2$$

where $F' = f$. This means that the "energy" of u is decreasing with time.

(ii) A stationary solution $U(x)$ of

$$\begin{cases} -\Delta U &= f(U) \qquad \text{in} \quad \Omega \\ U &= 0 \qquad \text{on} \quad \partial\Omega \end{cases}$$

is called *asymptotically stable* if there is an $\varepsilon > 0$ such that for any $u_\circ \in C(\overline{\Omega})$ satisfying $u_\circ = 0$ on $\partial\Omega$ and

$$\|u_\circ - U\|_{L^\infty(\Omega)} < \varepsilon$$

we have that the solution $u(x,t)$ of

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& f(u) & \text{in} \quad \Omega \times (0,T) \\ u &=& 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) &=& u_\circ(x) & \text{for} \quad t = 0 \end{array} \right. \tag{0.1}$$

exists for all time (that is, $T = \infty$) and

$$\lim_{t\to\infty} u(x,t) = U(x) \qquad \text{uniformly for } x \in \Omega.$$

Prove that if $U \in C^2(\overline{\Omega})$ is an asymptotically stable solution then

$$\mathcal{E}(U) \le \mathcal{E}(U+\eta) \quad \text{for all } \eta \in C_c^\infty(\Omega) \quad \text{with} \quad \|\eta\|_{L^\infty(\Omega)} < \varepsilon,$$

where $\mathcal{E}(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - F(w)\right)$.

(4 points)

(4) Let $u \in C(\overline{\Omega} \times [0,T])$ be a solution of

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& -u^2 & \text{in} \quad \Omega \times (0,T] \\ u &=& 0 & \text{on} \quad \partial\Omega \times (0,T]. \end{array} \right.$$

Prove that $u(x,T) \le 1/T$, regardless of the initial data at $t = 0$.

Hint: Use the comparison principle.

(3 points)

(5) The KPP equation

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& u(1-u) & \text{in} \quad \mathbb{R}^n \times (0,T) \\ u(x,0) &=& u_\circ(x) & \text{for} \quad t = 0 \end{array} \right.$$

is one of the most classical reaction-diffusion PDEs, and models population dynamics.

(i) Prove that if $0 < u_\circ(x) < 1$ for all $x \in \Omega$ then $0 < u(x,t) < 1$ for all $t > 0$, $x \in \Omega$.

(ii) Prove that for any $e \in \mathbb{S}^{n-1}$ there is a travelling-wave solution of the type

$$u(x,t) = v(x \cdot e - ct), \qquad v(z) = \frac{1}{(1+e^{\beta z})^2}$$

for some $\beta > 0$ and some $c > 0$.

(3 points)

(6) Using that any solution $U(x,t)$ to the heat equation

$$\left\{ \begin{array}{rcll} \partial_t U - \Delta U &=& 0 & \text{in} \quad \Omega \times (0,\infty) \\ U &=& 0 & \text{on} \quad \partial\Omega \times (0,\infty) \end{array} \right.$$

satisfies, for $1 \le p \le q \le \infty$,

$$\|U(t)\|_{L^q(\Omega)} \le C t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|U(0)\|_{L^p(\Omega)},$$

prove that:

(i) Given a C^∞ and globally Lipschitz function f, and any initial data $u_\circ \in L^1(\Omega)$, there exists a solution to the nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u &= f(u) & \text{in} \quad \Omega \times (0,T) \\ u &= 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) &= u_\circ(x) & \text{for} \quad t = 0. \end{cases}$$

for a short time $T > 0$.

(ii) Such solution $u(t)$ is bounded (and therefore C^∞) for positive times $t > 0$.

(4 points)

(7) Let $\vec{w} \in C^\infty(\overline{\Omega})$ be given.

(i) Prove that there exist functions $\vec{w_\circ} : \Omega \to \mathbb{R}^n$ and $q : \Omega \to \mathbb{R}$, such that

$$\vec{w} = \vec{w_\circ} + \nabla q$$

$$\operatorname{div} \vec{w_\circ} = 0 \text{ in } \Omega$$

and with $q = 0$ on $\partial\Omega$.

(ii) Prove that such representation for $\vec{w}$ is unique. Thus, we may denote

$$\vec{w_\circ} = \Pi\vec{w},$$

the Leray projection of $\vec{w}$.

(iii) Using the results from Chapter 2, deduce that

$$\|\Pi\vec{w}\|_{H^k(\Omega)} \leq C\|\vec{w}\|_{H^k(\Omega)}$$

for all $k \geq 1$.

(4 points)

(8) Let $\vec{u} \in C^\infty(\overline{\Omega} \times (0,T))$ be a solution of the Navier-Stokes equations

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} &= \Delta\vec{u} - \nabla p & \text{in} \quad \Omega \times (0,T) \\ \operatorname{div} \vec{u} &= 0 & \text{in} \quad \Omega \times (0,T) \\ \vec{u} &= 0 & \text{on} \quad \partial\Omega \times (0,T) \end{cases} \tag{0.2}$$

Prove that

$$\frac{d}{dt}\int_\Omega |\vec{u}|^2 dx \leq 0$$

(3 points)

(9) Let $\vec{u} \in C^\infty(\overline{\Omega})$ and $\Omega \subset \mathbb{R}^2$ (that is, $n = 2$).

Prove that if $\operatorname{div} \vec{u} = 0$ in Ω, then there exists a unique function ψ such that

$$\vec{u} = \operatorname{curl} \psi := (-\partial_{x_2}\psi, \, \partial_{x_1}\psi)$$

and $\psi = 0$ on $\partial\Omega$.

Moreover, prove that ψ

$$-\Delta\psi = \omega \quad \text{in} \quad \Omega.$$

where $\omega := \partial_{x_1} u_2 - \partial_{x_2} u_1$.

(2 points)

(10) Let $\Omega \subset \mathbb{R}^2$ (in 2D), and $\vec{u} \in C^\infty(\overline{\Omega} \times [0,T))$ be any solution of the Navier-Stokes equation (0.2).

(i) Prove that, if we denote $\vec{u} = (u_1, u_2)$, then the vorticity

$$\omega(x,t) := \operatorname{curl} \vec{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1,$$

solves the PDE

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = \Delta \omega \quad \text{in} \quad \Omega \times (0,T).$$

Prove also that

$$\Delta \vec{u} = (-\partial_{x_2}\omega, \partial_{x_1}\omega) \quad \text{in} \quad \Omega,$$

so that, since $\vec{u} = 0$ on $\partial\Omega$, then $\vec{u}$ is uniquely determined by ω.

(ii) Assuming that $\int_{\partial\Omega} \omega \frac{\partial \omega}{\partial \nu} \le 0$ for all $t > 0$, prove that

$$\|\omega(t)\|_{L^2(\Omega)} \le \|\omega_\circ\|_{L^2(\Omega)} \qquad \text{for all } t \in (0,T),$$

where $\omega_\circ$ is the vorticity at time $t = 0$.

(iii) Using the previous Exercise, and the results from Chapter 2, prove that

$$\|\vec{u}(t)\|_{H^1(\Omega)} \le C\|\omega_\circ\|_{L^2(\Omega)}.$$

This means that, in 2D, the H^1 norm of the solution cannot blow-up in finite time.

(4 points)

(11) Let us consider the Navier-Stokes equation (0.2) in dimension $n \le 3$.

Using the bound for the heat equation

$$\|P_t[v]\|_{H^2(\Omega)} \le \begin{cases} Mt^{-1/2}\|v\|_{H^1(\Omega)} & \text{for} \quad t \in (0,1) \\ Me^{-\lambda_1 t}\|v\|_{H^1(\Omega)} & \text{for} \quad t \ge 1, \end{cases}$$

prove that there exists $\delta > 0$ such that if the initial data satisfies

$$\|\vec{u}_\circ\|_{H^2(\Omega)} < \delta$$

then the solution $\vec{u}$ of (0.2) exists for all time $t > 0$ (that is, $T = \infty$).

Note: Use without proof, as in the proof of short-time existence for (0.2), the inequality $\|(\vec{w} \cdot \nabla)\vec{w}\|_{H^1(\Omega)} \le C\|\vec{w}\|^2_{H^2(\Omega)}$, valid for all functions $\vec{w} \in H^2(\Omega)$ with $\vec{w} = 0$ on $\partial\Omega$.

(3 points)